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A Unit Root Test Based on the Modified Least Squares Estimator

(Ujian Unit Akar Berdasarkan Penganggar Ubah Suai Kuasa Dua Terkecil)

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ABSTRACT

A unit root test based on the modified least squares (MLS) estimator for first-order autoregressive process is proposed and compared with unit root tests based on the ordinary least squares (OLS), the weighted symmetric (WS) and the modified weighted symmetric (MWS) estimators. The percentiles of the null distributions of the unit root test are also reported. The empirical probabilities of type I error and powers of the unit root tests were estimated via Monte Carlo simulation. The simulation results showed that all unit root tests can control the probability of type I error for all situations. The empirical power of the $\hat{\kappa}_{mws}$ *test is higher than the other unit root tests,* $\hat{\kappa}_{ols}, \hat{\kappa}_{ws}$ *and* $\hat{\kappa}_{mls}$*. Apart from that, the* $\hat{\tau}_{ws}$ *and* $\hat{\tau}_{mws}$ *tests also provide the highest empirical power. As an illustration, the monthly series of U.S. nominal interest rates on three-month treasury bills is analyzed.*

Keywords: First-order autoregressive; ordinary least squares estimator; unit root test; weighted symmetric estimator

ABSTRAK

Suatu ujian unit akar berdasarkan anggaran ubah suai kuasa dua terkecil (MLS) untuk proses autoregrasi peringkat pertama yang dicadang dan dibandingkan dengan penganggaran ujian unit akar yang berasaskan kuasa dua terkecil biasa (OLS), dengan wajaran simetri (WS) dan yang wajaran simetri ubah suai (MWS). Peratusan taburan nol ujian unit akar juga dilaporkan. Kebarangkalian empirikal daripada jenis ralat I dan kuasa ujian unit akar dianggarkan melalui simulasi Monte Carlo. Keputusan simulasi menunjukkan bahawa semua ujian unit akar boleh mengawal kemungkinan jenis ralat I untuk semua keadaan. Kuasa empirikal ujian $\hat{\kappa}_{mws}$ *adalah lebih tinggi daripada lain-lain ujian unit akar seperti* $\hat{\kappa}_{ols}, \hat{\kappa}_{ws}$ *dan* $\hat{\kappa}_{mls}$*. Selain itu, dalam ujian* $\hat{\tau}_{ws}$ *dan* $\hat{\tau}_{mws}$ *juga memberikan kuasa empirikal yang tertinggi. Sebagai ilustrasi, siri bulanan kadar faedah nominal US pada bil perbendaharaan tiga bulan dianalisis.*

Kata kunci: Autoregrasi peringkat pertama; penganggar kuasa dua terkecil biasa; penganggar wajaran simetri; ujian unit akar

INTRODUCTION

In time series analysis, Hamilton (1994) described the econometrical applications for the first-order autoregressive process. He also discussed the necessity of using the unit root test in order to find the correct model for the nominal interest rate and real GNP of the United States from the period of 1947 to 1989. These time series were plotted in Figures 17.2 and 17.3 of Hamilton (1994). Hamilton (1994) indicated that there are no guarantees in economic theory to suggest that nominal interest rates should exhibit a deterministic time trend, although Figure 17.2 displays an upward trend over the sample data. Consequently, the model for these time series may be a random walk process without trend or a stationary process with a constant term. In order to answer this question, the unit root test can be applied to select between these two processes. The unit root test has drawn much attention for the past three decades, especially in economics and other related fields. Statisticians and econometricians are interested in the unit root test since economic time series data may be non-stationary. Contributions to the unit root literature include the works of Fuller (1976, 1996), Dickey and Fuller (1979, 1981), Said and Dickey (1984, 1985), Phillips (1987), Phillips and Perron (1988), Hall (1989), Pantula and Hall (1991), Lucas (1995), Park (2003), Paparoditis and Politis (2005), among others.

The first-order autoregressive process $\{y_t, t = 1,2,\ldots,n\}$ denoted as AR(1) is given by

$$y_t = \delta + \rho y_{t-1} + e_t, \tag{1}$$

where $\delta = \mu(1 - \rho)$, μ is the mean of the process, ρ is the autoregressive coefficient and e_t are a sequence of independent and identically random variables from a normal distribution with zero mean and variance σ_e^2. Defining the AR polynomial by $\rho(z) = 1 - \rho z$, we can rewrite the process as

$$\rho(B)(y_t - \mu) = e_t,$$

where B is the backward shift operator such that $B^k y_t = y_{t-k}$. Equation (1) is called a stationary AR(1) process if and only if the root of the AR characteristic equation ($\rho(z) = 0$) exceeds 1 in absolute value, i.e., $|\rho| < 1$, otherwise it is called a non-stationary process or random walk process. In the case of a near non-stationary process, i.e., $|\rho|\rightarrow 1$,the mean and variance of this process change over time. The null hypothesis; H_0 and the alternative hypothesis; H_a for the unit root tests are as follows:

$$H_0 : \rho = 1$$

and

$$H_a : \rho < 1.$$

A common feature of almost all unit root tests is that they make use of the ordinary least squares (OLS) estimator. Although the OLS estimator has asymptotic normality for $|\rho|<1$ (Brockwell & Davis 1991; Mann & Wald 1943), it has long been known that the OLS estimator can have large bias and is sensitive to the occurrence of outliers in the data; for example, Marriott and Pope (1954), Newbold and Agiakloglou (1993) and Shaman and Stine (1988). There have been useful improvements in parameter estimation so as to reduce the bias of the OLS estimator. Denby and Martin (1979) presented the robust estimator for an autoregressive model. Gonzalez-Farias and Dickey (1992) considered maximum likelihood (ML) estimation for the parameters of the autoregressive process and suggested tests for unit roots on the basis of these estimators. Park and Fuller (1995) proposed the weighted symmetric (WS) estimator of an autoregressive parameter. Fuller (1996) presented a modification of the weighted symmetric (MWS) estimator. Shin and So (1999) developed an adaptive maximum likelihood procedure. Guo (2000) developed the simple and robust estimator for the AR(1) model. A modification for the least squares (MLS) estimator for a first-order autoregressive process in the case of the unit root was proposed by Youssef (2008). However, Youssef (2008) does not develop the testing for a unit root based on the modified least squares estimator. Thus, the main idea of the paper is to develop a unit root test based on the estimator proposed by Youssef (2008) and to evaluate the efficiency of four unit root tests based on OLS, WS, MLS and MWS estimators.

The organization of the paper is as follows. In the next section, we explain the details of the estimators and unit root tests. The performance of the unit root test based on the modified least squares estimator is examined and compared with those of some other unit root tests through Monte Carlo experiments in the section that follow. Next, we illustrate a real data example of the unit root tests proposed in this paper. The final section is devoted to conclusion.

DETAILED DESCRIPTION OF THE ESTIMATORS AND UNIT ROOT TESTS

The ordinary least square (OLS) estimator for (δ, ρ), can be obtained by regressing y_t on y_{t-1} as in (1). So, these estimators are given by,

$$\hat{\rho}_{ols} = \frac{\sum_{t=2}^{n}\left(y_{t-1} - \overline{y}_{(-1)}\right) y_t}{\sum_{t=2}^{n}\left(y_{t-1} - \overline{y}_{(-1)}\right)^2}, \tag{2}$$

and

$$\hat{\delta}_{ols} = \overline{y}_{(0)} - \hat{\rho}_{ols}\overline{y}_{(-1)}, \tag{3}$$

where

$$\overline{y}_{(0)} = (n-1)^{-1}\sum_{t=2}^{n} y_t \text{ and } \overline{y}_{(-1)} = (n-1)^{-1}\sum_{t=2}^{n} y_{t-1}.$$

Park and Fuller (1995) proposed the weighted symmetric (WS) estimator of (δ, ρ) given by

$$\hat{\rho}_{ws} = \frac{\sum_{t=2}^{n}(y_{t-1} - \overline{y})(y_t - \overline{y})}{\sum_{t=2}^{n-1}(y_t - \overline{y})^2 + n^{-1}\sum_{t=1}^{n}(y_t - \overline{y})^2}, \tag{4}$$

and

$$\hat{\delta}_{ws} = \overline{y}\left(1 - \rho^*_{ws}\right), \tag{5}$$

where

$$\overline{y} = n^{-1}\sum_{t=1}^{n} y_t \quad \text{and} \quad \rho^*_{ws} = \begin{cases} \hat{\rho}_{ws} & \text{if } |\hat{\rho}_{ws}| < 1, \\ 1 & \text{if } \hat{\rho}_{ws} \geq 1, \\ -1 & \text{if } \hat{\rho}_{ws} \leq -1. \end{cases}$$

Youssef (2008) suggested a modification to the OLS that is approximated for $\rho \in (-1,\infty)$ as follows

$$\hat{\rho}_{mls} = \hat{\rho}_{ols} + c\left(\hat{\tau}_{ols}\right)\left[\hat{V}\left(\hat{\rho}_{ols}\right)\right]^{1/2}, \tag{6}$$

and

$$\hat{\delta}_{mls} = \overline{y}_{(0)} - \hat{\rho}_{mls}\overline{y}_{(-1)}, \tag{7}$$

where

$$c\left(\hat{\tau}_{ols}\right) = \begin{cases} 0 & \text{if } \hat{\tau}_{ols} < -7.1, \\ 0.062222\left(\hat{\tau}_{ols} + 7.1\right)^2 & \text{if } -7.1 \leq \hat{\tau}_{ols} < -3.6, \\ 1.71 - 0.062222\left(\hat{\tau}_{ols} + 0.10\right)^2 & \text{if } -3.6 \leq \hat{\tau}_{ols} \leq 3.4, \\ 0.062222\left(\hat{\tau}_{ols} - 6.90\right)^2 & \text{if } 3.4 < \hat{\tau}_{ols} \leq 6.9, \\ 0 & \text{if } \hat{\tau}_{ols} > 6.9 \end{cases}$$

and

$$\hat{\tau}_{ols} = \left[\hat{V}\left(\hat{\rho}_{ols}\right)\right]^{-1/2}\left(\hat{\rho}_{ols} - 1\right).$$

Fuller (1996) introduced a modification of the weighted symmetric (MWS) estimator for $\rho \in (-1,1)$ given by

$$\hat{\rho}_{mws} = \hat{\rho}_{ws} + c\left(\hat{\tau}_{ws}\right)\left[\hat{V}\left(\hat{\rho}_{ws}\right)\right]^{1/2}, \tag{8}$$

and

$$\hat{\delta}_{mws} = \overline{y}\left(1-\hat{\rho}_{mws}\right), \quad (9)$$

where

$$c\left(\hat{\tau}_{ws}\right) = \begin{cases} -\hat{\tau}_{ws} & \text{if } \hat{\tau}_{ws} \geq -1.2, \\ 0.035672\left(\hat{\tau}_{ws}+7.0\right)^2 & \text{if } -7.0 < \hat{\tau}_{ws} < -1.2, \\ 0 & \text{if } \hat{\tau}_{ws} \leq -7.0 \end{cases}$$

and

$$\hat{\tau}_{ws} = \left[\hat{V}\left(\hat{\rho}_{ws}\right)\right]^{-1/2}\left(\hat{\rho}_{ws}-1\right).$$

The Dickey and Fuller unit root tests associated with $\hat{\rho}_{ols}$ are $\hat{\kappa}_{ols}$ and $\hat{\tau}_{ols}$ where

$$\hat{\kappa}_{ols} = n\left(\hat{\rho}_{ols}-1\right), \quad (10)$$

and

$$\hat{\tau}_{ols} = \left[\hat{V}\left(\hat{\rho}_{ols}\right)\right]^{-1/2}\left(\hat{\rho}_{ols}-1\right), \quad (11)$$

and $\hat{V}\left(\hat{\rho}_{ols}\right)$ is the estimated variance of $\hat{\rho}_{ols}$ defined as

$$\hat{V}\left(\hat{\rho}_{ols}\right) = \frac{\hat{\sigma}^2_{ols}}{\sum_{t=2}^{n}\left(y_{t-1}-\overline{y}_{(-1)}\right)^2},$$

where

$$\hat{\sigma}^2_{ols} = \left(n-3\right)^{-1}\sum_{t=2}^{n}\left(y_{t-1}-\hat{y}_t\right)^2,$$

and

$$\hat{y}_t = \hat{\delta}_{ols} + \hat{\rho}_{ols}y_{t-1}.$$

Counterparts of $\hat{\kappa}_{ols}$ and $\hat{\tau}_{ols}$, unit root tests based on the weighted symmetric (WS) estimator are $\hat{\kappa}_{ws}$ and $\hat{\tau}_{ws}$ where

$$\hat{\kappa}_{ws} = n\left(\hat{\rho}_{ws}-1\right), \quad (12)$$

and

$$\hat{\tau}_{ws} = \left[\hat{V}\left(\hat{\rho}_{ws}\right)\right]^{-1/2}\left(\hat{\rho}_{ws}-1\right), \quad (13)$$

and $\hat{V}\left(\hat{\rho}_{ws}\right)$ is the estimated variance of $\hat{\rho}_{ws}$ defined as

$$\hat{V}\left(\hat{\rho}_{ws}\right) = \frac{\hat{\sigma}^2_{ws}}{\sum_{t=2}^{n-1}\left(y_t-\overline{y}\right)^2 + n^{-1}\sum_{t=1}^{n}\left(y_t-\overline{y}\right)^2},$$

where

$$\hat{\sigma}^2_{ws} = \left(n-2\right)^{-1}Q\left(\hat{\rho}_{ws}\right),$$

$$Q\left(\hat{\rho}_{ws}\right) = \sum_{t=2}^{n} w_t\left(\left(y_t-\overline{y}\right)-\hat{\rho}_{ws}\left(y_{t-1}-\overline{y}\right)\right)^2 + \sum_{t=1}^{n-1}\left(1-w_{t+1}\right)\left(\left(y_t-\overline{y}\right)-\hat{\rho}_{ws}\left(y_{t+1}-\overline{y}\right)\right)^2,$$

and

$$w_t = n^{-1}(t-1).$$

Similar to $\hat{\kappa}_{ws}$ and $\hat{\tau}_{ws}$, unit root tests based on the modified least square (MLS) estimator are $\hat{\kappa}_{mls}$ and $\hat{\tau}_{mls}$ where

$$\hat{\kappa}_{mls} = n\left(\hat{\rho}_{mls}-1\right), \quad (14)$$

and

$$\hat{\tau}_{mls} = \left[\hat{V}\left(\hat{\rho}_{mls}\right)\right]^{-1/2}\left(\hat{\rho}_{mls}-1\right), \quad (15)$$

and $\hat{V}\left(\hat{\rho}_{mls}\right)$ is the estimated variance of $\hat{\rho}_{mls}$ defined as

$$\hat{V}\left(\hat{\rho}_{mls}\right) = \frac{\hat{\sigma}^2_{mls}}{\sum_{t=2}^{n}\left(y_{t-1}-y_{(-1)}\right)^2},$$

where

$$\hat{\sigma}^2_{mls} = \left(n-3\right)^{-1}\sum_{t=2}^{n}\left(y_{t-1}-\hat{y}_t\right)^2,$$

and

$$\hat{y}_t = \hat{\delta}_{mls} + \hat{\rho}_{mls}y_{t-1}.$$

Analogues of $\hat{\kappa}_{mls}$ and $\hat{\tau}_{mls}$, unit root tests based on the modified weighted symmetric (MWS) estimator are $\hat{\kappa}_{mws}$ and $\hat{\tau}_{mws}$ where

$$\hat{\kappa}_{mws} = n\left(\hat{\rho}_{mws}-1\right), \quad (16)$$

and

$$\hat{\tau}_{mws} = \left[\hat{V}\left(\hat{\rho}_{mws}\right)\right]^{-1/2}\left(\hat{\rho}_{mws}-1\right), \quad (17)$$

and $\hat{V}\left(\hat{\rho}_{mws}\right)$ is the estimated variance of $\hat{\rho}_{mws}$ defined as

$$\hat{V}\left(\hat{\rho}_{mws}\right) = \frac{\hat{\sigma}^2_{mws}}{\sum_{t=2}^{n-1}\left(y_t-\overline{y}\right)^2 + n^{-1}\sum_{t=1}^{n}\left(y_t-\overline{y}\right)^2},$$

where

$$\hat{\sigma}^2_{mws} = \left(n-2\right)^{-1}Q\left(\hat{\rho}_{mws}\right),$$

$$Q\left(\hat{\rho}_{mws}\right) = \sum_{t=2}^{n} w_t\left(\left(y_t-\overline{y}\right)-\hat{\rho}_{mws}\left(y_{t-1}-\overline{y}\right)\right)^2 + \sum_{t=1}^{n-1}\left(1-w_{t+1}\right)\left(\left(y_t-\overline{y}\right)-\hat{\rho}_{mws}\left(y_{t+1}-\overline{y}\right)\right)^2,$$

TABLE 1. Percentiles of the null distributions of the $\hat{\kappa}_{ols}$, $\hat{\kappa}_{ws}$, $\hat{\kappa}_{mws}$ and $\hat{\kappa}_{mls}$ tests

Sample	Probability that $\hat{\kappa}_{ols}$ is less than entry							
Sizes n	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
25	-17.17	-14.59	-12.49	-10.24	-0.75	0.00	0.64	1.40
50	-18.75	-15.67	-13.23	-10.71	-0.80	-0.07	0.52	1.22
100	-19.66	-16.29	-13.66	-10.98	-0.83	-0.11	0.46	1.12
250	-20.15	-16.63	-13.90	-11.15	-0.84	-0.14	0.43	1.07
500	-20.41	-16.77	-13.99	-11.20	-0.84	-0.14	0.43	1.07
Sample	Probability that $\hat{\kappa}_{ws}$ is less than entry							
Sizes n	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
25	-16.88	-14.21	-12.04	-9.70	-0.15	0.49	1.01	1.64
50	-18.11	-14.98	-12.48	-9.91	-0.07	0.56	1.07	1.68
100	-18.83	-15.39	-12.74	-10.04	-0.05	0.58	1.08	1.67
250	-19.16	-15.60	-12.87	-10.11	-0.02	0.60	1.10	1.67
500	-19.30	-15.69	-12.92	-10.13	-0.02	0.59	1.09	1.66
Sample	Probability that $\hat{\kappa}_{mls}$ is less than entry							
Sizes n	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
25	-12.62	-9.42	-7.10	-4.85	3.03	3.89	4.70	5.72
50	-13.28	-9.85	-7.37	-5.00	2.83	3.61	4.31	5.17
100	-13.66	-10.08	-7.53	-5.09	2.72	3.46	4.13	4.94
250	-13.85	-10.22	-7.62	-5.15	2.67	3.39	4.03	4.80
500	-13.96	-10.28	-7.64	-5.16	2.65	3.37	4.00	4.77
Sample	Probability that $\hat{\kappa}_{mws}$ is less than entry							
Sizes n	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
25	-14.49	-11.40	-8.96	-6.37	0.00	0.00	0.00	0.00
50	-15.24	-11.80	-9.10	-6.37	0.00	0.00	0.00	0.00
100	-15.68	-11.99	-9.19	-6.40	0.00	0.00	0.00	0.00
250	-15.85	-12.07	-9.24	-6.41	0.00	0.00	0.00	0.00
500	-15.92	-12.12	-9.26	-6.41	0.00	0.00	0.00	0.00

The probability shown at the head of the column is the area in the left-hand tail

and

$$w_t = n^{-1}(t-1).$$

The percentiles of the null distributions of the unit root tests are shown in Tables 1 and 2. These values are based on the average of percentiles of the 100 sets of percentiles from 10000 independent simulated test statistics.

MONTE CARLO EXPERIMENTS

In this section, we describe the results of several Monte Carlo experiments carried out to evaluate the performance of the unit root tests, $\hat{\kappa}_{ols}$, $\hat{\kappa}_{ws}$, $\hat{\kappa}_{mls}$, $\hat{\kappa}_{mws}$, $\hat{\tau}_{ols}$, $\hat{\tau}_{ws}$, $\hat{\tau}_{mls}$ and $\hat{\tau}_{mws}$. The first-order autoregressive process in (1) with parameters $(\mu, \sigma_e) = (0, 1)$ is generated under the null hypothesis H_0 by setting $y_0 \sim N\left(0, \frac{1}{1-\rho^2}\right)$ and the initial one hundred observations are generated and discarded in order to eliminate the effect of the initial value. The scope of the simulations is set under the autoregressive parameter values $\rho = 0.70, 0.80, 0.85, 0.9, 0.93, 0.95, 0.97, 0.98, 0.99$ and 1.00; the sample sizes $n = 25, 50, 100$ and 250. The random variables e_t are generated from a normal distribution with a mean of zero and variance of one. Two hundred thousand time series were simulated by using R statistical software (Ihaka & Gentleman 1996; The R Development Core Team 2012a, 2012b). The significance levels α for the unit root tests are equal to 0.01, 0.05 and 0.10. The simulations compared the empirical probability of type I error and power of the unit root tests. The simulation results are summarized in Tables 3 to 6. We begin with the results for the probability of type I error of the unit root test (Table 3). The Bradley's (1978) criteria were considered.

TABLE 2. Percentiles of the null distributions of the $\hat{\tau}_{ols}, \hat{\tau}_{ws}, \hat{\tau}_{mws}$ and $\hat{\tau}_{mls}$ tests

Sample Sizes n	Probability that $\hat{\tau}_{ols}$ is less than entry							
	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
25	-3.73	-3.32	-2.99	-2.63	-0.37	0.01	0.33	0.72
50	-3.57	-3.22	-2.92	-2.60	-0.41	-0.04	0.28	0.65
100	-3.50	-3.17	-2.89	-2.58	-0.42	-0.06	0.26	0.63
250	-3.46	-3.14	-2.87	-2.57	-0.44	-0.07	0.25	0.61
500	-3.44	-3.13	-2.87	-2.57	-0.44	-0.08	0.24	0.61
Sample Sizes n	Probability that $\hat{\tau}_{ws}$ is less than entry							
	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
25	-3.39	-2.99	-2.67	-2.32	-0.07	0.25	0.53	0.86
50	-3.24	-2.89	-2.60	-2.27	-0.04	0.30	0.59	0.93
100	-3.17	-2.84	-2.56	-2.25	-0.02	0.33	0.63	0.97
250	-3.13	-2.81	-2.54	-2.24	-0.01	0.33	0.64	0.99
500	-3.12	-2.80	-2.54	-2.23	-0.01	0.34	0.64	0.99
Sample Sizes n	Probability that $\hat{\tau}_{mls}$ is less than entry							
	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
25	-2.99	-2.20	-1.75	-1.28	1.26	1.61	1.91	2.25
50	-2.62	-2.08	-1.68	-1.26	1.26	1.62	1.92	2.26
100	-2.49	-2.03	-1.65	-1.24	1.26	1.63	1.93	2.27
250	-2.44	-2.00	-1.64	-1.24	1.26	1.63	1.94	2.28
500	-2.42	-1.99	-1.63	-1.24	1.26	1.63	1.94	2.28
Sample Sizes n	Probability that $\hat{\tau}_{mws}$ is less than entry							
	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
25	-2.91	-2.40	-1.98	-1.52	0.00	0.00	0.00	0.00
50	-2.73	-2.28	-1.90	-1.47	0.00	0.00	0.00	0.00
100	-2.64	-2.21	-1.85	-1.44	0.00	0.00	0.00	0.00
250	-2.60	-2.18	-1.83	-1.43	0.00	0.00	0.00	0.00
500	-2.58	-2.18	-1.82	-1.42	0.00	0.00	0.00	0.00

The probability shown at the head of the column is the area in the left-hand tail

TABLE 3. Probabilities of type I error of the $\hat{\kappa}_{ols}, \hat{\kappa}_{ws}, \hat{\kappa}_{mls}, \hat{\kappa}_{mws}, \hat{\tau}_{ols}, \hat{\tau}_{ws}, \hat{\tau}_{mls}$ and $\hat{\tau}_{mws}$

α	n	$\hat{\kappa}_{ols}$	$\hat{\kappa}_{ws}$	$\hat{\kappa}_{mls}$	$\hat{\kappa}_{mws}$	$\hat{\tau}_{ols}$	$\hat{\tau}_{ws}$	$\hat{\tau}_{mls}$	$\hat{\tau}_{mws}$
0.01	25	0.01006	0.01000	0.00997	0.01031	0.00957	0.01024	0.00999	0.01002
	50	0.00961	0.00977	0.00981	0.00974	0.00957	0.00986	0.00998	0.00926
	100	0.00981	0.01000	0.01006	0.00992	0.00983	0.01044	0.01021	0.01025
	250	0.00989	0.00992	0.00994	0.00993	0.01001	0.00998	0.01003	0.01017
0.05	25	0.05075	0.05052	0.05097	0.05099	0.04927	0.05039	0.05112	0.05054
	50	0.05025	0.05019	0.05020	0.05045	0.04986	0.05008	0.05068	0.05090
	100	0.05051	0.05023	0.05037	0.05019	0.05085	0.05088	0.04983	0.05098
	250	0.04893	0.05004	0.05030	0.04882	0.04878	0.05057	0.05091	0.05027
0.10	25	0.09951	0.09956	0.09916	0.09857	0.10071	0.09893	0.09917	0.09919
	50	0.10010	0.10012	0.09995	0.09977	0.09946	0.10112	0.10134	0.10045
	100	0.10004	0.10026	0.10072	0.10015	0.10109	0.10185	0.09965	0.10136
	250	0.10019	0.10032	0.10026	0.09987	0.10035	0.10210	0.09987	0.09942

TABLE 4. Empirical powers of the $\hat{\kappa}_{ols}, \hat{\kappa}_{ws}, \hat{\kappa}_{mls}, \hat{\kappa}_{mws}, \hat{\tau}_{ols}, \hat{\tau}_{ws}, \hat{\tau}_{mls}$ and $\hat{\tau}_{mws}$ when $\alpha = 0.01$

n	ρ	$\hat{\kappa}_{ols}$	$\hat{\kappa}_{ws}$	$\hat{\kappa}_{mls}$	$\hat{\kappa}_{mws}$	$\hat{\tau}_{ols}$	$\hat{\tau}_{ws}$	$\hat{\tau}_{mls}$	$\hat{\tau}_{mws}$
25	0.70	0.08955	0.09371	0.08141	0.09554	0.05078	0.09758	0.05284	0.09571
	0.80	0.04250	0.04433	0.03944	0.04494	0.02568	0.04580	0.02676	0.04474
	0.85	0.02876	0.02950	0.02702	0.02991	0.01881	0.03065	0.01971	0.03004
	0.90	0.02025	0.02075	0.01974	0.02092	0.01459	0.02125	0.01526	0.02070
	0.93	0.01549	0.01558	0.01514	0.01576	0.01193	0.01574	0.01241	0.01538
	0.95	0.01400	0.01391	0.01398	0.01404	0.01133	0.01406	0.01183	0.01374
	0.97	0.01209	0.01210	0.01203	0.01201	0.01086	0.01218	0.01127	0.01185
	0.98	0.01121	0.01124	0.01092	0.01137	0.01019	0.01131	0.01074	0.01098
	0.99	0.01070	0.01062	0.01084	0.01068	0.01005	0.01085	0.01050	0.01055
50	0.70	0.41792	0.45496	0.38302	0.46130	0.27164	0.46098	0.26479	0.46250
	0.80	0.15937	0.17579	0.14397	0.17882	0.09571	0.18025	0.09272	0.18148
	0.85	0.08220	0.09006	0.07505	0.09136	0.04952	0.09252	0.04752	0.09322
	0.90	0.04120	0.04389	0.03850	0.04444	0.02630	0.04497	0.02551	0.04532
	0.93	0.02715	0.02839	0.02571	0.02861	0.01913	0.02890	0.01838	0.02920
	0.95	0.01924	0.01979	0.01863	0.02000	0.01460	0.02002	0.01402	0.02029
	0.97	0.01470	0.01505	0.01435	0.01506	0.01219	0.01496	0.01161	0.01512
	0.98	0.01270	0.01288	0.01226	0.01294	0.01063	0.01290	0.01022	0.01297
	0.99	0.01116	0.01117	0.01108	0.01126	0.01053	0.01124	0.01010	0.01133
100	0.70	0.97989	0.98740	0.97244	0.98750	0.92807	0.98643	0.93172	0.98572
	0.80	0.68908	0.73792	0.65499	0.74239	0.51586	0.74718	0.52593	0.74216
	0.85	0.38846	0.43198	0.36014	0.43723	0.25455	0.44633	0.26150	0.44086
	0.90	0.15120	0.16893	0.13860	0.17233	0.09218	0.17842	0.09547	0.17551
	0.93	0.07295	0.07961	0.06732	0.08070	0.04461	0.08410	0.04648	0.08242
	0.95	0.04109	0.04423	0.03861	0.04502	0.02699	0.04693	0.02822	0.04594
	0.97	0.02241	0.02352	0.02159	0.02382	0.01623	0.02473	0.01687	0.02425
	0.98	0.01684	0.01745	0.01640	0.01765	0.01299	0.01844	0.01362	0.01802
	0.99	0.01280	0.01281	0.01265	0.01276	0.01111	0.01319	0.01155	0.01289
250	0.70	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
	0.80	1.00000	1.00000	1.00000	1.00000	0.99997	1.00000	0.99997	1.00000
	0.85	0.99883	0.99953	0.99805	0.99947	0.99177	0.99923	0.99204	0.99923
	0.90	0.87740	0.91332	0.85226	0.91437	0.75117	0.91260	0.75423	0.91293
	0.93	0.51552	0.57276	0.47924	0.57658	0.36335	0.57991	0.36684	0.58089
	0.95	0.24143	0.27516	0.21986	0.27795	0.15563	0.28201	0.15758	0.28261
	0.97	0.07954	0.08897	0.07283	0.08963	0.05134	0.09136	0.05205	0.09171
	0.98	0.04013	0.04383	0.03721	0.04411	0.02706	0.04503	0.02755	0.04521
	0.99	0.01879	0.01951	0.01798	0.01962	0.01511	0.01988	0.01536	0.01994

TABLE 5. Empirical powers of the $\hat{\kappa}_{ols}, \hat{\kappa}_{ws}, \hat{\kappa}_{mls}, \hat{\kappa}_{mws}, \hat{\tau}_{ols}, \hat{\tau}_{ws}, \hat{\tau}_{mls}$ and $\hat{\tau}_{mws}$ when $\alpha = 0.05$

n	ρ	$\hat{\kappa}_{ols}$	$\hat{\kappa}_{ws}$	$\hat{\kappa}_{mls}$	$\hat{\kappa}_{mws}$	$\hat{\tau}_{ols}$	$\hat{\tau}_{ws}$	$\hat{\tau}_{mls}$	$\hat{\tau}_{mws}$
25	0.70	0.31271	0.33700	0.28504	0.34298	0.21062	0.34186	0.21526	0.34596
	0.80	0.17558	0.18648	0.16204	0.18991	0.12111	0.18908	0.12422	0.19227
	0.85	0.12722	0.13494	0.11877	0.13714	0.09128	0.13657	0.09362	0.13883
	0.90	0.09284	0.09678	0.08866	0.09764	0.07138	0.09710	0.07332	0.09869
	0.93	0.07514	0.07692	0.07300	0.07762	0.06138	0.07737	0.06296	0.07853
	0.95	0.06635	0.06772	0.06522	0.06804	0.05766	0.06744	0.05929	0.06852
	0.97	0.05938	0.06022	0.05877	0.06022	0.05322	0.05920	0.05459	0.06030
	0.98	0.05555	0.05635	0.05556	0.05654	0.05132	0.05571	0.05275	0.05664
	0.99	0.05325	0.05289	0.05322	0.05286	0.05086	0.05236	0.05223	0.05316
50	0.70	0.80273	0.84538	0.76542	0.84827	0.65079	0.84379	0.65644	0.84521
	0.80	0.47309	0.51894	0.43557	0.52649	0.33481	0.52564	0.33963	0.52849
	0.85	0.30615	0.33742	0.28011	0.34285	0.20945	0.34339	0.21305	0.34581
	0.90	0.17547	0.19111	0.16218	0.19426	0.12197	0.19467	0.12438	0.19631
	0.93	0.12138	0.12912	0.11382	0.13078	0.08957	0.13142	0.09133	0.13263
	0.95	0.09368	0.09727	0.08928	0.09816	0.07286	0.09823	0.07444	0.09915
	0.97	0.07123	0.07307	0.06936	0.07389	0.06027	0.07391	0.06172	0.07457
	0.98	0.06337	0.06434	0.06146	0.06458	0.05546	0.06417	0.05641	0.06482
	0.99	0.05616	0.05659	0.05571	0.05703	0.05195	0.05672	0.05297	0.05720
100	0.70	0.99968	0.99990	0.99934	0.99985	0.99722	0.99971	0.99725	0.99969
	0.80	0.95408	0.97325	0.93651	0.97275	0.87767	0.97041	0.87811	0.96895
	0.85	0.78339	0.83726	0.74524	0.83930	0.63787	0.83831	0.63863	0.83381
	0.90	0.46133	0.51526	0.42499	0.51959	0.33323	0.52448	0.33386	0.51813
	0.93	0.27039	0.30389	0.24850	0.30703	0.19021	0.31232	0.19067	0.30744
	0.95	0.17452	0.19233	0.16145	0.19448	0.12536	0.19757	0.12562	0.19404
	0.97	0.10507	0.11145	0.09978	0.11247	0.08263	0.11483	0.08282	0.11273
	0.98	0.08147	0.08515	0.07790	0.08536	0.06768	0.08667	0.06780	0.08468
	0.99	0.06226	0.06385	0.06133	0.06392	0.05704	0.06501	0.05720	0.06350
250	0.70	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
	0.80	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
	0.85	1.00000	1.00000	1.00000	1.00000	0.99990	1.00000	0.99992	1.00000
	0.90	0.99386	0.99753	0.99007	0.99723	0.96978	0.99623	0.97161	0.99625
	0.93	0.88095	0.92285	0.85245	0.92284	0.75725	0.91998	0.76433	0.92066
	0.95	0.61856	0.68490	0.57901	0.68954	0.46888	0.69157	0.47703	0.69308
	0.97	0.29296	0.33102	0.27024	0.33508	0.20774	0.33925	0.21290	0.34055
	0.98	0.17004	0.18875	0.15826	0.19176	0.12261	0.19475	0.12609	0.19572
	0.99	0.09187	0.09715	0.08729	0.09806	0.07298	0.09937	0.07495	0.09997

TABLE 6. Empirical powers of the $\hat{\kappa}_{ols}, \hat{\kappa}_{ws}, \hat{\kappa}_{mls}, \hat{\kappa}_{mws}, \hat{\tau}_{ols}, \hat{\tau}_{ws}, \hat{\tau}_{mls}$ and $\hat{\tau}_{mws}$ when $\alpha = 0.10$

n	ρ	$\hat{\kappa}_{ols}$	$\hat{\kappa}_{ws}$	$\hat{\kappa}_{mls}$	$\hat{\kappa}_{mws}$	$\hat{\tau}_{ols}$	$\hat{\tau}_{ws}$	$\hat{\tau}_{mls}$	$\hat{\tau}_{mws}$
25	0.70	0.49797	0.53755	0.45078	0.54517	0.37101	0.54459	0.36694	0.54613
	0.80	0.31249	0.33669	0.28301	0.34209	0.23257	0.34332	0.22926	0.34453
	0.85	0.23608	0.25123	0.21664	0.25494	0.18170	0.25606	0.17919	0.25711
	0.90	0.17689	0.18490	0.16543	0.18683	0.14306	0.18715	0.14121	0.18779
	0.93	0.14759	0.15212	0.14109	0.15311	0.12665	0.15301	0.12471	0.15360
	0.95	0.13059	0.13327	0.12614	0.13395	0.11696	0.13399	0.11520	0.13467
	0.97	0.11725	0.11891	0.11495	0.11921	0.10955	0.11887	0.10797	0.11938
	0.98	0.11107	0.11144	0.10937	0.11156	0.10649	0.11161	0.10492	0.11216
	0.99	0.10575	0.10598	0.10471	0.10594	0.10391	0.10547	0.10240	0.10595
50	0.70	0.92809	0.95360	0.89868	0.95261	0.83258	0.95020	0.83470	0.95015
	0.80	0.67424	0.73143	0.62475	0.73555	0.53111	0.73636	0.53400	0.73674
	0.85	0.48565	0.53533	0.44353	0.54206	0.36492	0.54589	0.36739	0.54660
	0.90	0.30977	0.33845	0.28306	0.34312	0.23185	0.34724	0.23389	0.34803
	0.93	0.22470	0.24139	0.20864	0.24439	0.17484	0.24734	0.17633	0.24795
	0.95	0.17699	0.18634	0.16687	0.18814	0.14381	0.19062	0.14502	0.19095
	0.97	0.14027	0.14486	0.13470	0.14570	0.12274	0.14739	0.12393	0.14788
	0.98	0.12435	0.12677	0.12067	0.12702	0.11257	0.12848	0.11350	0.12874
	0.99	0.11008	0.11077	0.10893	0.11041	0.10600	0.11182	0.10692	0.11192
100	0.70	0.99997	1.00000	0.99993	0.99999	0.99962	0.99997	0.99963	0.99997
	0.80	0.99182	0.99640	0.98462	0.99560	0.96347	0.99450	0.96367	0.99396
	0.85	0.91951	0.95158	0.88928	0.95049	0.82204	0.94800	0.82254	0.94555
	0.90	0.66404	0.73011	0.61641	0.73547	0.52621	0.73746	0.52697	0.73103
	0.93	0.44095	0.49506	0.40388	0.50181	0.33357	0.50747	0.33415	0.49985
	0.95	0.30796	0.34138	0.28258	0.34688	0.23370	0.35170	0.23426	0.34572
	0.97	0.19809	0.21454	0.18569	0.21708	0.15975	0.22047	0.16009	0.21625
	0.98	0.15689	0.16438	0.14969	0.16642	0.13307	0.16876	0.13344	0.16539
	0.99	0.12487	0.12797	0.12199	0.12858	0.11464	0.13040	0.11491	0.12754
250	0.70	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
	0.80	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
	0.85	1.00000	1.00000	1.00000	1.00000	0.99999	1.00000	0.99999	1.00000
	0.90	0.99947	0.99990	0.99874	0.99973	0.99497	0.99955	0.99490	0.99951
	0.93	0.96829	0.98450	0.95112	0.98309	0.90785	0.98117	0.90655	0.98014
	0.95	0.81137	0.86828	0.76905	0.86913	0.68298	0.86922	0.68059	0.86455
	0.97	0.47710	0.53491	0.43883	0.54118	0.36483	0.54839	0.36267	0.54091
	0.98	0.30612	0.34149	0.28235	0.34621	0.23583	0.35291	0.23415	0.34668
	0.99	0.17691	0.18880	0.16812	0.19060	0.14684	0.19497	0.14565	0.19094

This criteria is that if the empirical probability of type I error of any unit root test is within the interval 0.5α and 1.5α, then that unit root test can control the probability of type I error. For the significance level $\alpha = 0.05$, the empirical probability of type I error should be between 0.025 and 0.075. It can be seen in Table 3 that all unit root tests can control the probability of type I error for all sample sizes and all levels of significance. The empirical probability of type I error for all unit root tests gets closer to the significance level with increasing sample sizes n. This is intuitive in nature because as n increases it is possible to estimate the autoregressive coefficients more accurately. Tables 4 to 6 show that the empirical power of the $\hat{\kappa}_{mws}$ test is higher than the other unit root test, $\hat{\kappa}_{ols}$,$\hat{\kappa}_{ws}$ and $\hat{\kappa}_{mls}$. Furthermore, the $\hat{\tau}_{ws}$ and $\hat{\tau}_{mws}$ tests provide the highest empirical power. Apart from that, the empirical power of the tests tends to increase as the sample size gets larger. On the other hand, the empirical power of the tests decreases when ρ approaches unity as the AR(1) process becomes less distinguishable from random walks.

REAL DATA EXAMPLE

In this section, we illustrate empirically the use of all the unit root tests studied in this paper. Data on monthly U.S nominal interest rates on three-month treasury bills from Economagic (http://www.economagic.com) are applied to construct the unit root tests. The data, plotted in Figure 1, span 1981:1-1989:12. The data consist of 108 observations. In addition, the sample autocorrelation function (ACF) and sample partial autocorrelation function (PACF) are shown in Figure 2. For assistance in interpreting these functions, two-standard-error limits are plotted on the graphs as dashed lines. From considering Figure 2, the autocorrelations decay exponentially to zero, while the partial autocorrelations cut off after one lag. Therefore, from this evidence, an appropriate model for this time series is an AR(1) model. The optimal models fitted to this series based on OLS, WS, MLS and MWS estimators are

OLS: $\hat{Y}_t = 0.3880+0.9470\,\hat{Y}_{t-1}$ with $\hat{V}(\hat{\rho}_{ols}) = 0.00052$,

WS: $\hat{Y}_t = 0.1172+0.9863\,\hat{Y}_{t-1}$ with $\hat{V}(\hat{\rho}_{ws}) = 0.00056$,

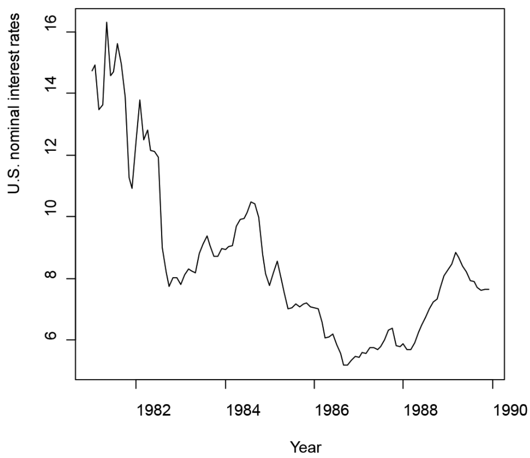

FIGURE 1. Monthly U.S. nominal interest rates on three-month treasury bills

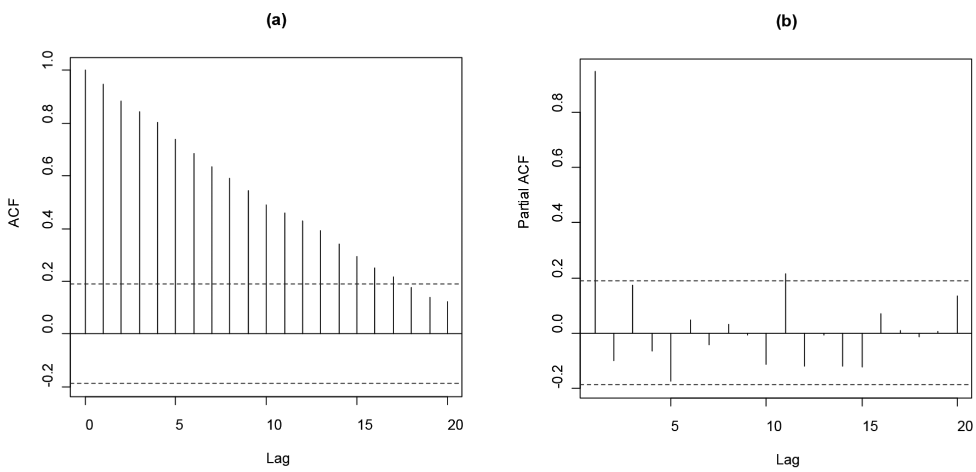

FIGURE 2. (a) The sample ACF and (b) the sample PACF, for the monthly U.S. nominal interest rates on three-month treasury bills

TABLE 7. The unit root test statistics based on OLS, WS, MLS and MWS estimators and the results of hypothesis testing for monthly U.S. nominal interest rates on three-month treasury bills

Methods	Statistics $\hat{\kappa}$	Results	Statistics $\hat{\tau}$	Results
OLS	$\hat{\kappa}_{ols} = -5.73$	Not reject H_0	$\hat{\tau}_{ols} = -2.32$	Not reject H_0
WS	$\hat{\kappa}_{ws} = -1.48$	Not reject H_0	$\hat{\tau}_{ws} = -0.57$	Not reject H_0
MLS	$\hat{\kappa}_{mls} = -2.27$	Not reject H_0	$\hat{\tau}_{mls} = -0.91$	Not reject H_0
MWS	$\hat{\kappa}_{mws} = 0.00$	Not reject H_0	$\hat{\tau}_{mws} = 0.00$	Not reject H_0

MLS: $\hat{Y}_t = 0.1139+0.9790\,\hat{Y}_{t-1}$ with $\hat{V}(\hat{\rho}_{mls}) = 0.00053$,

MWS: $\hat{Y}_t = \hat{Y}_{t-1}$ with $\hat{V}(\hat{\rho}_{mws}) = 0.00056$.

For the testing of the unit root, the null hypothesis; H_0 and the alternative hypothesis; H_a are $H_0 : \rho = 1$ and $H_a : \rho < 1$. The unit root test statistics based on OLS, WS, MLS and MWS estimators and the results of hypothesis testing are shown in Table 7.

As can be seen from Table 7, the values of the unit root test statistics are less than the critical values in Tables 1 and 2 at significance level $\alpha = 0.05$. Therefore, the null hypothesis; H_0 is not rejected, i.e., $\rho = 1$. We conclude that this time series data is non-stationary.

CONCLUSION

This paper proposes a new unit root test based on the modified least squares estimator of Youssef (2008). Four unit root tests based on the ordinary least squares (OLS), the weighted symmetric (WS), the modified least squares (MLS) and the modified weighted symmetric (MWS) estimators were studied and compared by examining the empirical probabilities of type I error and powers of the tests. The tables of critical values for testing of the unit root are created by setting $\rho = 1$. Based on simulation studies, all unit root tests can control the probability of type I error for all situations. The empirical power of the $\hat{\kappa}_{mws}$ test is higher than the other unit root tests, $\hat{\kappa}_{ols}, \hat{\kappa}_{ws}$ and $\hat{\kappa}_{mls}$ and the $\hat{\tau}_{ws}$ and $\hat{\tau}_{mws}$ tests provide the highest empirical power. The performance of the testing of the unit root is illustrated with a real data example.

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